A NOTE ON THE VERY AMPLENESS OF COMPLETE LINEAR SYSTEMS ON BLOWINGS-UP OF $\mathbb{P}^3$

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Abstract. In this note we consider the blowing-up X of $\mathbb{P}^3$ along r general points of the anticanonical divisor of a smooth quadric in $\mathbb{P}^3$. Given a complete linear system $\mathcal{L} = |dH - m_1E_1 - \cdots - m_rE_r|$ on X, with H the pull-back of a plane in $\mathbb{P}^3$ and E_i the exceptional divisor corresponding to P_i, we give necessary and sufficient conditions for the very ampleness (resp. base point freeness and non-speciality) of $\mathcal{L}$. As a corollary we obtain a sufficient condition for the very ampleness of such a complete linear system on the blowing-up of $\mathbb{P}^3$ along r general points.

1. Introduction

In this note we work over an algebraically closed field of characteristic 0.

Let $P_1, \ldots, P_r$ be general points of the anticanonical divisor of a smooth quadric in $\mathbb{P}^3$ and choose some integers $m_1 \geq \ldots \geq m_r \geq 0$. Consider the linear system $\mathcal{L}'$ of surfaces of degree d in $\mathbb{P}^3$ having multiplicities at least m_i at P_i, for all $i = 1, \ldots, r$. Let X denote the blowing-up of $\mathbb{P}^3$ alongs $P_1, \ldots, P_r$, and let $\mathcal{L}$ denote the complete linear system on X corresponding to $\mathcal{L}'$.

and the system is called *special* if $\dim(\mathcal{L}) > \max\{-1, \mathrm{vdim}\,(\mathcal{L})\}$.

Let Z be a zero-dimensional subscheme of length 2 of X, then $\mathcal{L}$ separates Z if there exists a divisor $D \in \mathcal{L}$ such that $Z \cap D \neq \emptyset$ but $Z \not\subset D$. The system $\mathcal{L}$ on X is called very ample if it separates all such Z.

The very ampleness of line bundles of blowings-up of varieties has been studied by several authors, e.g. Stéphane Chauvin and the first author [2], Marc Coppens [3, 4], Brian Harbourne [8], Mauro C. Beltrametti and Andrew J. Sommese [1].

In theorem 5.1 (resp. theorem 4.1) we prove that such a system $\mathcal{L}$ is very ample (resp. base point free) on X if and only if $m_r > 0$, $d \geq m_1 + m_2 + 1$ and $4d \geq m_1 + \cdots + m_r + 3$ (resp. $d \geq m_1 + m_2$ and $4d \geq m_1 + \cdots + m_r + 2$). A fundamental tool for proving these results is theorem 3.1 which states that a system $\mathcal{L}$ with $2d \geq m_1 + \cdots + m_4$ is non-special if $d \geq m_1 + m_2 - 1$ and $4d \geq m_1 + \cdots + m_r$.

If $r \leq 8$, the points P_i are in general position on $\mathbb{P}^3$ and the dimension and base locus of $\mathcal{L}$ on X can be determined using the results from [5, 6].

The techniques used in this note are a generalization of the ones in [5, 6] and make use of the results about complete linear systems on rational surfaces with irreducible anticanonical divisor (see [7, 8]).

2. Preliminaries and notation

Let $\mathcal{L}_3(d)$ denote the complete linear system of surfaces of degree d in $\mathbb{P}^3$. Consider a general quadric $\bar{Q} \in \mathcal{L}_3(2)$ in $\mathbb{P}^3$ and let $K_{\bar{Q}}$ denote the canonical class on $\bar{Q}$. Then we know that $-K_{\bar{Q}}$ is just the linear system on $\bar{Q}$ induced by $\mathcal{L}_3(2)$, so we can consider $D_{\bar{Q}} \in -K_{\bar{Q}}$ which is smooth and irreducible.

Let $P_1, \ldots, P_r$ be general points of $D_{\bar{Q}}$ and choose integers $m_1 \geq \cdots \geq m_r \geq 0$. By X_r we denote the blowing-up of $\mathbb{P}^3$ along the points $P_1, \ldots, P_r$, E_0 denotes the pullback of a plane in $\mathbb{P}^3$, by E_i $(i = 1, \ldots, r)$ we mean the exceptional divisor on X_r corresponding to P_i and $\pi : X_r \to \mathbb{P}^3$ denotes the projection map.

On $\mathbb{P}^3$, we let $\mathcal{L}_3(d; m_1, \ldots, m_r)$ denote the linear system of surfaces of degree d with multiplicities at least m_i at P_i for all $i = 1, \ldots, r$ as well as the corresponding sheaf.

By abuse of notation, on X_r, $\mathcal{L}_3(d; m_1, \ldots, m_r)$ also denotes the invertible sheaf $\pi^*(\mathcal{O}_{\mathbb{P}^3}(d)) \otimes \mathcal{O}_{X_r}(-m_1E_1 - \cdots - m_rE_r)$ and the corresponding complete linear system $|dE_0 - m_1E_1 - \cdots m_rE_r|$.

Analoguously, on $\mathbb{P}^3$, $\mathcal{L}_3(d; m_1^{n_1}, \ldots, m_t^{n_t})$ denotes the linear system of surfaces of degree d with multiplicities at least m_i at n_i of the points on D_Q as well as the corresponding sheaf. Again, the same notation is used to denote the associated complete linear system and invertible sheaf on X_r.

The virtual dimension of the linear system $\mathcal{L} = \mathcal{L}_3(d; m_1, \ldots, m_r)$ on $\mathbb{P}^3$ as well as on X_r is defined as

$$\mathrm{vdim}\,(\mathcal{L}) := \binom{d+3}{3} - \sum_{i=1}^{r} \binom{m_i+2}{3} - 1.$$

The expected dimension of $\mathcal{L}$ is then given by

$$\mathrm{edim}\,(\mathcal{L}) := \max\{-1, \mathrm{vdim}\,(\mathcal{L})\}.$$

It is then clear that $\dim(\mathcal{L}) \geq \mathrm{edim}\,(\mathcal{L}) \geq \mathrm{vdim}\,(\mathcal{L})$, and the system $\mathcal{L}$ is called *special* if $\dim(\mathcal{L}) > \mathrm{edim}\,(\mathcal{L})$. The system $\mathcal{L}$ is associated to the sections of the sheaf $\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z$, where $Z = \sum m_i p_i$ is the zero-dimensional scheme of fat points. From the cohomology exact sequence associated to

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

we obtain that $h^i(\mathcal{L}) = 0$ for $i = 2, 3$. Therefore $v(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) - 1$, so that a non-empty system is special if and only if $h^1(\mathcal{L}) > 0$.

Note that the strict transform Q_r of $\bar{Q}$ on X_r is a divisor of $\mathcal{L}_3(2; 1^r)$, and Q_r is just the blowing-up of $\bar{Q}$ along $P_1, \ldots, P_r$. So $\mathrm{Pic}\,(Q_r) = \langle f_1, f_2, e_1, \ldots, e_r \rangle$, with f_1 and f_2 the pullbacks of the two rulings on $\bar{Q}$ and $e_1, \ldots, e_r$ the exceptional curves. By $\mathcal{L}_{Q_r}(a, b; m_1, \ldots, m_r)$ we denote the complete linear system $|af_1 + bf_2 - m_1e_1 - \ldots - m_re_r|$, and, as before, if some of the multiplicities are the same, we also use the notation $\mathcal{L}_{Q_r}(a, b; m_1^{n_1}, \ldots, m_t^{n_t})$.

Let B_s be the blowing-up of $\mathbb{P}^2$ along s general points of a smooth irreducible cubic, then $\mathrm{Pic}\, B_s = \langle h, e_1', \ldots, e_s' \rangle$, with h the pullback of a line and e_l' the exceptional curves. By $\mathcal{L}_2(d; m_1, \ldots, m_s)$ we denote the complete linear system $|dh - m_1e_1' - \ldots - m_se_s'|$. And again, as before, if some of the multiplicities are the same, we

also use the notation $\mathcal{L}_2(d; m_1^{n_t}, \ldots, m_r^{n_t})$. Note that $-K_{B_s} = \mathcal{L}_2(3; 1^s)$ contains a smooth irreducible divisor which we will denote by D_{B_s}.

On B_s, a system $\mathcal{L}_2(d; m_1, \ldots, m_s)$ is said to be in standard form if $d \geq m_1+m_2+m_3$ and $m_1 \geq m_2 \geq \cdots \geq m_s \geq 0$; and it is called standard it there exists a base $\langle \tilde{h}, \tilde{e}_1, \ldots, \tilde{e}_s \rangle$ of $\operatorname{Pic} B_s$ such that $\mathcal{L}_2(d; m_1, \ldots, m_s) = |\tilde{d}\tilde{h} - \tilde{m}_1\tilde{e}_1 - \ldots - \tilde{m}_s\tilde{e}_s|$ is in standard form.

As explained in [5, §6], the blowing-up Q of the quadric along 1 general point can also be seen as a blowing-up of the projective plane along 2 general points, and

$$\mathcal{L}_Q(a, b; m) = \mathcal{L}_2(a + b - m; a - m, b - m).$$

So, in particular $-K_Q = \mathcal{L}_Q(2, 2; 1) = \mathcal{L}_2(3; 1^2) = -K_{B_2}$. Obviously, this means that our blowing-up Q_r can also be seen as a B_{r+1} and

$$\mathcal{L}_{Q_r}(a, b; m_1, m_2, \ldots, m_r) = \mathcal{L}_2(a + b - m_1; a - m_1, b - m_1, m_2, \ldots, m_r).$$

This implies in particular that we can apply the results from [7] and [8].

3. Non-speciality

Theorem 3.1. *Consider $\mathcal{L} = \mathcal{L}_3(d; m_1, \ldots, m_r)$ on X_r with $2d \geq m_1 + m_2 + m_3 + m_4$ and $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. Then $h^1(\mathcal{L}) = 0$ if*

(1) *$d \geq m_1 + m_2 - 1$ and*
(2) *$4d \geq m_1 + \cdots m_r$ if $r \geq 9$.*

Proof. We will assume that $m_r > 0$, since otherwise we can work on $X_{r'}$ with $r' := \max\{i : m_i > 0\}$. If $r \leq 8$ then the points $P_1, \ldots, P_r$ are general points of $\mathbb{P}^3$ and the statement follows from [5, Theorem 5.3]. So we assume that $r \geq 9$ and consider the following exact sequence

$$0 \longrightarrow \mathcal{L}_3(d - 2; m_1 - 1, \ldots, m_r - 1) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_{Q_r} \longrightarrow 0$$

Then $\mathcal{L} \otimes \mathcal{O}_{Q_r} = \mathcal{L}_{Q_r}(d, d; m_1, \ldots, m_r) = \mathcal{L}_2(2d - m_1; (d - m_1)^2, m_2, \ldots, m_r)$. Proceeding as in the proof of [5, Lemma 5.2] it is easily seen that this is a standard class. Moreover $\mathcal{L}_2(2d-m_1; (d-m_1)^2, m_2, \ldots, m_r).K_{B_{r+1}} = -4d+m_1+\cdots+m_r \leq 0$, so we can apply [7, Theorem 1.1 and Proposition 1.2] to obtain $h^1(\mathcal{L}\otimes\mathcal{O}_{Q_r}) = 0$. On the other hand, one can easily check that $\mathcal{L}' := \mathcal{L}_3(d - 2; m_1 - 1, \ldots, m_r - 1)$ still satisfies the conditions of the theorem. Continuing like this until the residue class $\mathcal{L}' = \mathcal{L}_3(d'; m_1', \ldots, m_{r'}')$ is such that $r' \leq 8$. For this class we then know that $h^1(\mathcal{L}') = 0$ which gives us that $h^1(\mathcal{L}) = 0$. □

4. Base point freeness

Theorem 4.1. *Consider $\mathcal{L} = \mathcal{L}_3(d; m_1, \ldots, m_r)$ on X_r with $m_1 \geq m_2 \geq \cdots \geq m_r$. Then $\mathcal{L}$ is base point free on X_r if and only if the following conditions are satisfied*

(1) *$m_r \geq 0$,*
(2) *$d \geq m_1 + m_2$ and*
(2) *$4d \geq m_1 + \cdots m_r + 2$ if $r \geq 8$.*

Proof. First of all let us prove that the conditions are necessary. Obviously, if $m_r < 0$ then $m_r E_r \subset \mathrm{Bs}(\mathcal{L})$; and if $d < m_1 + m_2$ then the strict transform of the line through P_1 and P_2 is contained in $\mathrm{Bs}(\mathcal{L})$. Now assume that (1) and (2) are satisfied, but $4d \leq m_1 + \cdots m_r + 1$ and consider

$$0 \longrightarrow \mathcal{L}_3(d-2; m_1-1, \ldots, m_r-1) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_{Q_r} \longrightarrow 0 \qquad (1)$$

Proceeding as before, one can check that $\mathcal{L} \otimes \mathcal{O}_{Q_r} = \mathcal{L}_2(2d-m_1; (d-m_1)^2, m_2, \ldots, m_r)$ is standard and $\mathcal{L}_2(2d-m_1; (d-m_1)^2, m_2, \ldots, m_r).(-K_{B_{r+1}}) = 4d - m_1 - \cdots - m_r \leq 1$. Using the results from [7], we obtain that $\mathcal{L} \otimes \mathcal{O}_{Q_r}$ has base points, which will also be base points of $\mathcal{L}$.

Now assume that all three conditions are satisfied (as before, we may even assume $m_r > 0$). If $r \leq 8$, the result follows from [6, Theorem 6.2], so we assume that $r \geq 9$ and that the result holds for $r' < r$ Consider the exact sequence (1). As before, one can see that $\mathcal{L} \otimes \mathcal{O}_{Q_r} = \mathcal{L}_2(2d-m_1; (d-m_1)^2, m_2, \ldots, m_r)$ is standard and since $\mathcal{L}_2(2d-m_1; (d-m_1)^2, m_2, \ldots, m_r).K_{B_{r+1}} \leq -2$ we know that $\mathcal{L} \otimes \mathcal{O}_{Q_r}$ is base point free (see [7, Lemma 3.3(2)]). Also, because of theorem 3.1, we know that $h^1(\mathcal{L}_3(d-2; m_1-1, \ldots, m_r-1)) = 0$, so $\mathcal{L}$ induces the complete linear system $\mathcal{L} \otimes \mathcal{O}_{Q_r}$ on Q_r, which means in particular that $\mathcal{L}$ has no base points on Q_r. On the other hand it is easily checked that $\mathcal{L}_3(d-2; m_1-1, \ldots, m_r-1)$ still satisfies the conditions of the theorem. Continue like this untill you have $r' < r$ for the residue class (i.e. repeat this reasoning m_r times). Denote $\mathcal{L}_3(d-2m_r; m_1-m_r, \ldots, m_{r-1}-m_r)$ by $\mathcal{L}'$. We then know that $\mathcal{L}' + mQ_r \subset \mathcal{L}$. Since $\mathcal{L}$ has no base points on Q_r, and since $\mathcal{L}'$ is base point free by our induction hypothesis, we obtain that $\mathrm{Bs}(\mathcal{L}) = \emptyset$. □

5. Very ampleness

Theorem 5.1. *Consider $\mathcal{L} = \mathcal{L}_3(d; m_1, \ldots, m_r)$ on X_r with $m_1 \geq m_2 \geq \cdots \geq m_r$. Then $\mathcal{L}$ is very ample on X_r if and only if the following conditions are satisfied*

(1) $m_r > 0$,
(2) $d \geq m_1 + m_2 + 1$ *($d \geq m_1 + 1$ if $r = 1$; $d \geq 1$ if $r = 0$) and*
(2) $4d \geq m_1 + \cdots m_r + 3$ *if $r \geq 9$.*

Proof. First of all let us note that the conditions are necessary. Obviously, if $m_r \leq 0$ then $\mathcal{L}$ cannot separate on E_r; and if $d \leq m_1 + m_2$ then $\mathcal{L}$ cannot separate a zero-dimensional subscheme Z of length 2 of the strict transform of the line through P_1 and P_2. In case $4d \leq m_1 + \cdots m_r + 2$, one can see that $\mathcal{L}$ cannot separate Z if it is contained in D_{Q_r}.

Now assume that all three conditions are satisfied.

First of all consider the exact sequence

$$0 \longrightarrow \mathcal{L}_3(d; m_1+1, m_2, \ldots, m_r) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_{E_1} \longrightarrow 0$$

Because of Theorem 3.1 we know that $h^1(\mathcal{L}_3(d; m_1+1, m_2, \ldots, m_r)) = 0$, so $\mathcal{L}$ induces the complete linear system $\mathcal{L} \otimes \mathcal{O}_{E_1}$ on E_1. Since $\mathcal{L} \otimes \mathcal{O}_{E_1} = \mathcal{L}_2(m_1)$, we see that $\mathcal{L}$ separates on E_1. Naturally, a similar reasoning can be done for any E_i, so $\mathcal{L}$ separates on every E_i.

Moreover, one can easily check that $\mathcal{L}(E_i) := \mathcal{L}_3(d; m_1, \dots, m_{i-1}, m_i+1, m_{i+1}, \dots, m_r)$ satisfies all the conditions of Theorem 4.1, so that $\mathcal{L}(E_i)$ is base point free on X_r. This means that $\mathcal{L}$ can separate Z if $\exists i : Z \cap E_i \neq 0$ but $Z \not\subset E_i$.

Combining the previous two results, we see that we now only need to show that $\mathcal{L}$ separates Z with $Z \cap E_i = \emptyset$ for all $i = 1, \dots, r$.

In case $r = 0, 1$ or 2, this is trivial.

Now let us assume that $r \geq 3$ and that the statement holds for $r' < r$.

First look at the case where $m_r = 1$ and consider the exact sequence

$$0 \longrightarrow \mathcal{L}_3(d-2; m_1 - 1, \dots, m_{r-1} - 1, 0) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_{Q_r} \longrightarrow 0$$

Proceeding as before, one can easily see that $\mathcal{L} \otimes \mathcal{O}_{Q_r}$ is standard. Moreover, $4d - m_1 - \cdots - m_r \geq 3$ (if $r \geq 9$ this is condition (3) and if $3 \leq r \leq 8$ this follows from (2)), so [8, Theorem 2.1] implies that $\mathcal{L} \otimes \mathcal{O}_{Q_r}$ is very ample. Since $h^1(\mathcal{L}_3(d-2; m_1 - 1, \dots, m_{r-1} - 1)) = 0$ (because of Theorem 3.1) we then obtain that $\mathcal{L}$ separates on Q_r. Using Theorem 4.1 we also obtain that $\mathcal{L}' := \mathcal{L}_3(d - 2; m_1 - 1, \dots, m_{r-1} - 1, 0)$ is base point free, which implies that $\mathcal{L}$ separates Z if $Z \cap Q_r \neq \emptyset$. Let $r' = \max\{i : m_i > 1\}$ (or $r' = 0$ if all $m_i = 1$), then one can easily check that $\mathcal{L}'$ satisfies all the conditions of the theorem on $X_{r'}$. So, using the induction hypothesis, we have that $\mathcal{L}'$ is very ample on $X_{r'}$. But since Z on X_r is disjoint with all E_i, Z corresponds with a zero-dimensional subscheme on $X_{r'}$ (also disjoint with all E_i). So may may conclude that $\mathcal{L}$ separates any Z.

Now we assume $m_r > 1$ and we assume that the statement holds for $m_r' < m_r$. Consider the exact sequence

$$0 \longrightarrow \mathcal{L}_3(d-2; m_1 - 1, \dots, m_r - 1) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_{Q_r} \longrightarrow 0$$

Proceeding similarly as for the case $m_r = 1$ one can easily see that $\mathcal{L}$ separates all Z. $\square$

Remark 5.2. One can also use Theorem 4.1 and [1, Theorem 2.1] to obtain a very ampleness result. However in this way only part of the complete class of very ample systems on X_r are obtained.

Remark 5.3. Let $A_1, \dots, A_r$ be general points on $\mathbb{P}^3$, let Y_r be the blowing-up of $\mathbb{P}^3$ along those r general points and let $\mathcal{L}_3(d; m_1, \dots, m_r)$ ($m_1 \geq m_2 \geq \cdots \geq m_r$) denote the complete linear system $|dE_0 - m_1 E_1 - \cdots - m_r E_r|$ on Y_r. Since the very ampleness is an open property, Theorem 5.1 implies that $\mathcal{L}_3(d; m_1, \dots, m_r)$ is very ample on Y_r if $m_r > 0$, $d \geq m_1 + m_2 + 1$ ($d \geq m_1 + 1$ if $r = 1$; $d \geq 1$ if $r = 0$) and $4d \geq m_1 + \cdots m_r + 3$ if $r \geq 9$. Of course the third condition will now no longer be a necessary condition.

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